# Planar Topology: Connectedness and Paths

Open/Closed Sets, Connectedness, and Path-Connectedness

Lecture Notes

October 6, 2026

## 1 Interior, Exterior, Boundary, Closure

Let $K \subseteq \mathbb{C}$ (or $\widehat{\mathbb{C}}$, the extended plane); write $K^c = \mathbb{C} \setminus K$ (or $\widehat{\mathbb{C}} \setminus K$).

**Definition (neighborhood)**

$K$ is a **neighborhood** of $z_0 \in K$ if $\exists \delta > 0 : B(z_0, \delta) \subseteq K$.

**Definitions**

- **Interior**: $\operatorname{Int} K := \{z : \exists \delta > 0,\ B(z, \delta) \subseteq K\}$.
- **Exterior**: $\operatorname{Ext} K := \{z : \exists \delta > 0,\ B(z, \delta) \cap K = \varnothing\} = \operatorname{Int}(K^c)$.
- **Boundary**: $\partial K := (\operatorname{Int} K \cup \operatorname{Ext} K)^c = \partial(K^c) = \{z : \forall \delta > 0,\ B(z, \delta) \cap K \neq \varnothing \text{ and } B(z, \delta) \cap K^c \neq \varnothing\}$.
- **Limit points**: $L(K) := \{z : \exists\, z_n \in K,\ z_n \neq z,\ z_n \to z\}$.
- **Isolated points**: $I(K) := K \setminus L(K) = \{z \in K : \exists \delta > 0,\ B(z, \delta) \cap K = \{z\}\}$.
- **Closure**: $\operatorname{Clos} K := K \cup \partial K = (\operatorname{Ext} K)^c = K \cup L(K)$.

**Example.** $\operatorname{Int} B(z, \delta) = \operatorname{Int} \overline{B(z, \delta)} = B(z, \delta)$; $\operatorname{Clos} B(z, \delta) = \operatorname{Clos} \overline{B(z, \delta)} = \overline{B(z, \delta)}$; and $\partial B(z, \delta) = \{w : |w - z| = \delta\} = \partial \overline{B(z, \delta)}$.

## 2 Open and Closed Sets

**Definition**

$K$ is **open** if $K = \operatorname{Int} K$. $K$ is **closed** if $\partial K \subseteq K \iff K^c$ is open $\iff L(K) \subseteq K$.

**Properties**

1. If $(U_i)_{i \in I}$ is a family of open sets, $\bigcup_{i \in I} U_i$ is open, and $\bigcap_{i \in I} U_i$ is open if $I$ is finite.
2. If $(F_i)_{i \in I}$ is a family of closed sets, $\bigcap_{i \in I} F_i$ is closed, and $\bigcup_{i \in I} F_i$ is closed if $I$ is finite.
3. If $U$ is open and $U \subseteq K$, then $U \subseteq \operatorname{Int} K$.
4. If $F$ is closed and $K \subseteq F$, then $\operatorname{Clos} K \subseteq F$.

## 3 Connectedness

**Definition (connected)**

$K \subseteq \mathbb{C}$ (or $\widehat{\mathbb{C}}$, $\mathbb{R}$, ...) is **connected** if: whenever $K \subseteq U_1 \cup U_2$ with $U_1, U_2$ open and $U_1 \cap U_2 = \varnothing$, we have $K \subseteq U_1$ or $K \subseteq U_2$.

**Remark.** If $K$ is open, $K \cap U_1$ and $K \cap U_2$ are open, and the definition becomes: $K = (K \cap U_1) \cup (K \cap U_2)$ disjoint union of open sets $\implies$ one of them is empty.

**Theorem**

If $(K_i)_{i \in I}$ is a family of connected sets with $\bigcap_i K_i \neq \varnothing$, then $\bigcup_i K_i$ is connected.

**Proof.** Let $z_0 \in \bigcap_i K_i$, and suppose $\bigcup_i K_i \subseteq U_1 \cup U_2$ with $U_1, U_2$ open, disjoint. Say $z_0 \in U_1$. For each $j$, $K_j \subseteq U_1 \cup U_2$ and $z_0 \in U_1 \cap K_j \neq \varnothing$; since $K_j$ is connected, $K_j \subseteq U_1$. This holds for every $j$, so $\bigcup_i K_i \subseteq U_1$. □

**Theorem (connected components)**

Every $K \subseteq \mathbb{C}$ can be uniquely decomposed as $K = \bigcup_i K_i$, a disjoint union of nonempty connected sets $K_i$ (its **connected components**).

**Proof.** For $z \in K$, let $K_z := \bigcup \{F : z \in F \subseteq K,\ F \text{ connected}\}$. By the previous theorem, $K_z$ is connected (all the sets in the union share the point $z$) – it is the *maximal* connected subset of $K$ containing $z$. If $z \neq z'$, either $K_z \cap K_{z'} = \varnothing$, or $K_z \cap K_{z'} \neq \varnothing$, in which case $K_z \cup K_{z'}$ is connected (again by the previous theorem), so by maximality $K_z = K_{z'} = K_z \cup K_{z'}$. □

**Theorem (Generalized Intermediate Value Theorem)**

Let $K$ be connected and $f$ continuous on $K$. Then $f(K)$ is connected.

*Proof left as an exercise (use: $V \subseteq f(K)$ relatively open $\implies$ $f^{-1}(V)$ relatively open).*

## 4 Path-Connectedness

**Definitions (arc, path-connected)**

A continuous map $\gamma : [a, b] \to \mathbb{C}$ is called an **arc** (or **path**) from $\gamma(a)$ to $\gamma(b)$. $K \subseteq \mathbb{C}$ is **path-connected** if for all $z_1, z_2 \in K$ there is an arc $\gamma : [a, b] \to K$ from $z_1$ to $z_2$.

It is easy to check that $B(z, \delta)$ and $\overline{B(z, \delta)}$ are path-connected (join points by straight segments).

**Theorem**

Let $U \subseteq \mathbb{C}$ be open. Then $U$ is connected if and only if it is path-connected.

**Remark.** For *any* $K \subseteq \mathbb{C}$ (not necessarily open), path-connected $\implies$ connected.

**Proof of the Remark.** Suppose $K \subseteq U_1 \cup U_2$ (disjoint, open) meets both: $z_1 \in K \cap U_1$, $z_2 \in K \cap U_2$. Let $\gamma : [a, b] \to K$ be a path from $z_1$ to $z_2$, and let

$$t := \sup\{s \in [a, b] : \gamma(s) \in U_1\},$$

which is well defined since $\gamma(a) = z_1 \in U_1$. If $\gamma(t) \in U_2$, then by continuity $\gamma$ maps a neighborhood of $t$ into $U_2$ – but $t$ is a supremum of times with $\gamma(s) \in U_1$, so there are such $s$ arbitrarily close to $t$, a contradiction. So $\gamma(t) \in U_1$ (in particular $t < b$, since $\gamma(b) = z_2 \in U_2$); but then, by continuity again, $\gamma$ maps a neighborhood of $t$ into $U_1$, so $\gamma(s) \in U_1$ for some $s > t$ – contradicting that $t$ is the supremum. □

**Example (connected but not path-connected).** The "topologist's sine curve"

$$K = \left\{ \operatorname{Im} z = \sin \tfrac{1}{\operatorname{Re} z},\ 0 < \operatorname{Re} z \leq 1 \right\} \cup \{-1 \leq \operatorname{Im} z \leq 1,\ \operatorname{Re} z = 0\}$$

is connected, but there is no path from a point on the vertical segment to a point on the oscillating curve.

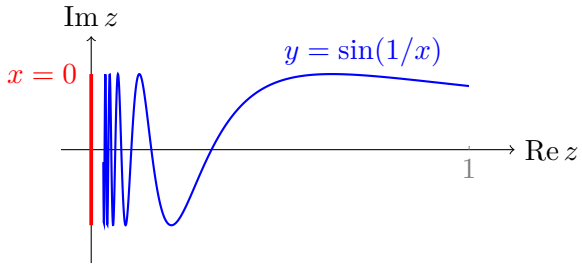



Figure 1: The topologist's sine curve: the graph of $y = \sin(1/x)$ for $0 < x \leq 1$ (blue), together with the vertical segment $\{x = 0,\ -1 \leq y \leq 1\}$ (red). The union is connected – the oscillating curve accumulates on the entire segment as $x \to 0^+$, so the segment cannot be separated off by disjoint open sets – but there is no path from a point on the segment to a point on the curve: any such path would have to oscillate infinitely often in a finite amount of "time," which is impossible for a continuous map from $[a, b]$.